pf sketch of Thm 7.2

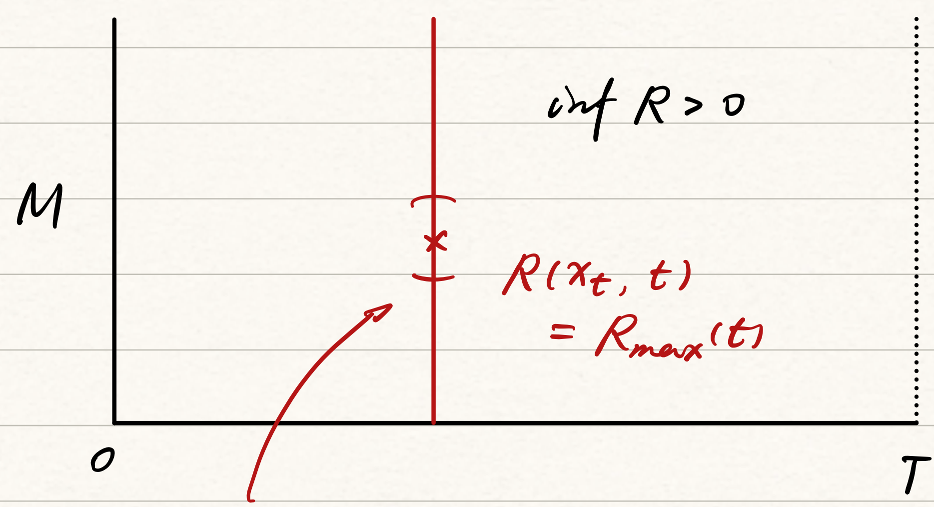
Step 1. Global pinching of R, $\displaystyle \lim_{t\to T} \frac{R_{max}}{R_{min}} = 1$

$$\inf_{M\times[0,T]} R > 0 \qquad \lim_{t\to T} R_{max}(t) = \infty$$

$$\exists\, C, \delta \quad \text{s.t.} \quad |\nabla R(x,t)| \le C\, R_{max}(t)^{\frac{3}{2}-\delta}$$

by Prop 7.4

fix t $\qquad R_{max}(t) \xrightarrow{t\to T} \infty$



$$B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{max}(t)}}\right)$$

on that ball

$$R_{max}(t) - R(x,t) \le \overbrace{\frac{1}{\eta\sqrt{R_{max}(t)}}}^{r} \cdot \max|\nabla R(t)|$$

$$\le \frac{C}{\eta} R_{max}(t)^{1-\delta}.$$

$$\Rightarrow \quad R(x,t) \geq R_{max}(t)\left(1 - \frac{C}{\eta} R_{max}(t)^{-\delta}\right).$$

Claim: the above ball is all of M.

pf. it follows by applying Thm 1.127

Theorem 1.127 (Bonnet-Myers). *If (M^n, g) is a complete Riemannian manifold with* $\mathrm{Rc} \geq (n-1)K$, *where* $K > 0$, *then* $\operatorname{diam}(g) \leq \pi/\sqrt{K}$. *In particular, M^n is compact and* $\pi_1(M) < \infty$.

and Prop 7.4.

Since $\lim_{t \to T} R_{max}(t) = \infty$, $\exists\ \tau \in [0, T)$ s.t.

$$\eta = \frac{C}{\eta} R_{max}(\tau)^{-\delta} \iff R_{max}(\tau) = \left(\frac{\eta^2}{C}\right)^{-\frac{1}{\delta}}$$

For $t \in [\tau, T)$ $\quad R(x,t) \geq R_{max}(t)(1-\eta)$

$x \in B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{max}(t)}}\right)$

Bonnet - Myers + pinching estimate $Rc \geq \varepsilon R g$

$\Rightarrow$ for $\eta > 0$ suff. small $\quad M = B_{g(t)}\left(x_t, \frac{1}{\eta\sqrt{R_{max}(t)}}\right)$

Step 2. estimate for NRF.

closed $(M, g(0))$, $Rc > 0$ $\exists\, C, \delta$ s.t.

NRF

$$|\widetilde{Rm} - \widetilde{\overset{\circ}{Rm}}| < C e^{-\delta \tilde{t}}$$

pf. scale invariant + Step 1 $\Rightarrow \lim_{\tilde{t} \to \tilde{T}} \frac{\tilde{R}_{max}}{\tilde{R}_{min}} = 1$

decay estimate of $\tilde{f} = \frac{|\widetilde{Rm} - \widetilde{\overset{\circ}{Rm}}|^2}{\tilde{R}^2}$

Part (6): Let

$$\tilde{f} \doteqdot \frac{\left|\widetilde{\mathrm{Rc}} - \frac{1}{3}\tilde{R}\tilde{g}\right|^2}{\tilde{R}^2}.$$

$\tilde{f}$ satisfies the same equation as for its counterpart $f \doteqdot \left|\mathrm{Rc} - \frac{1}{3}Rg\right|^2 / R^2$ for the unnormalized flow. This equation is the following (see Exercise 3.33 below):

$$\frac{\partial f}{\partial t} = \Delta f + 2\langle \nabla \log R, \nabla f\rangle - \frac{2}{R^4}\left|R\nabla_i R_{jk} - \nabla_i R R_{jk}\right|^2 + 4P, \tag{3.36}$$

≤ 0 if $Rc \geq \varepsilon R g$

where

$$P \doteqdot \frac{1}{R^3}\left(\frac{5}{2}R^2\left|\mathrm{Rc}\right|^2 - 2R\,\mathrm{Trace}_g\left(\mathrm{Rc}^3\right) - \frac{1}{2}R^4 - \left|\mathrm{Rc}\right|^4\right).$$

$\frac{\partial \tilde{f}}{\partial t} \leq \tilde{\Delta} \tilde{f} + \cdots$ and use max principle

Step 3. $|\tilde{\nabla}^k \widetilde{Rm}| \leq C e^{-\delta \tilde{t}}$

Lemma 3.36. *For any $k \in \mathbb{N}$ there exists a constant C depending only on k and n such that for any p-tensor $\alpha_{i_1 \cdots i_p}$*

(1)
$$\int_M \left|\nabla^j \alpha\right|^{2k/j} d\mu \leq C \max_M |\alpha|^{2\left(\frac{k}{j}-1\right)} \int_M \left|\nabla^k \alpha\right|^2 d\mu$$
for any $j = 1, \ldots, k-1$, and

(2)
$$\int_M \left|\nabla^j \alpha\right|^2 d\mu \leq C \left(\int_M \left|\nabla^k \alpha\right|^2 d\mu\right)^{j/k} \left(\int_M |\alpha|^2 d\mu\right)^{1-(j/k)}$$
for any $j = 0, \ldots, k$.

Proof. We first show that under the unnormalized Ricci flow

(3.40)
$$\frac{d}{dt} \int_M \left|\nabla^k \mathrm{Rm}\right|^2 d\mu \leq -2 \int_M \left|\nabla^{k+1} \mathrm{Rm}\right|^2 d\mu + C \max_M |\mathrm{Rm}| \int_M \left|\nabla^k \mathrm{Rm}\right|^2 d\mu$$
for some constant $C < \infty$. To prove this, by Exercise 6.29 we have
$$\frac{\partial}{\partial t} \left|\nabla^k \mathrm{Rm}\right|^2 \leq \Delta \left|\nabla^k \mathrm{Rm}\right|^2 - 2 \left|\nabla^{k+1} \mathrm{Rm}\right|^2 + \sum_{\ell=0}^{k} \left|\nabla^\ell \mathrm{Rm}\right| \left|\nabla^{k-\ell} \mathrm{Rm}\right| \left|\nabla^k \mathrm{Rm}\right|.$$

pf. Start with estimating RF.

$$\frac{d}{dt} \int |\nabla^k Rm|^2 d\mu + 2 \int_M |\nabla^{k+1} Rm|^2 d\mu$$

$$\leq \left(\int |\nabla^\ell Rm|^{\frac{2k}{\ell}} d\mu \right)^{\frac{\ell}{2k}} \cdot$$

$$\left(\int |\nabla^{k-\ell} Rm|^{\frac{2k}{k-\ell}} d\mu \right)^{\frac{k-\ell}{2k}} \cdot \left(\int |\nabla^k Rm|^2 d\mu \right)^{\frac{1}{2}}$$

works for $n \geq 4$, the arguement of Lemma 3.37 applies